A Geometric Framework for Registration of Sparse Images

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I. Introduction

Image registration is the process of aligning two or more images, taken from various viewpoints, at different times, or by different sensors. Many registration algorithms seek to solve the minimization problem:

$$(P) : \eta_0 = \underset{\eta \in \mathcal{T}}{\operatorname{argmin}} \|U(\eta)p - q\|_2,$$

where p and q are the input images, $(\mathcal{T}, \circ)$ defines a transformation group, and $U(\eta)p$ denotes the transformed version of p by η. Solving this optimization problem in all generality is hard since it involves the minimization of a non convex objective function [1].

We study in this work the problem (P) in the case where p and q are K-sparse in a redundant dictionary $\mathcal{D}$ that is obtained by transforming a non negative mother function ϕ with elements in $\mathcal{T}_d \subset \mathcal{T}$:

$$\mathcal{D} = \{U(\gamma)\phi : \gamma \in \mathcal{T}_d\}.$$

To prevent the mutual canceling of the features, we assume that coefficients of p and q are positive.

Our approach consists in solving a relaxed version $(\hat{P})$ of (P), where the search space $\mathcal{T}$ is replaced by the set of relative transformations between the features in both images. Specifically, if $p = \sum_{i=1}^{K} c_i \phi_{\gamma_i}$ and $q = \sum_{i=1}^{K} d_i \phi_{\delta_i}$, the problem $(\hat{P})$ is cast as follows:

$$(\hat{P}) : \hat{\eta} = \underset{\eta \in \mathcal{T}_a^{p,q}}{\operatorname{argmin}} \|U(\eta)p - q\|_2 ,$$

where $\mathcal{T}_a^{p,q} = \{\delta_i \circ \gamma_j^{-1} : 1 \leq i, j \leq K\}$. We can solve $(\hat{P})$ with a full search over the space $\mathcal{T}_a^{p,q}$ as long as K is relatively small.

Contributions: Our contribution is three-fold: (i) We propose a novel approach for image registration of sparse images in $\mathcal{D}$. (ii) We introduce two novel dictionary properties that characterize the registration performance. (iii) We examine the performance of the registration algorithm on illustrative examples [2].

II. Bound on the registration performance

We examine the penalty of relaxing the problem (P) into $(\hat{P})$. We measure the registration error with $E(p, q) = d_a(p, q) - d(p, q)$, where $d_a(p, q)$ and $d(p, q)$ are respectively equal to $\|U(\hat{\eta})p - q\|_2$ and $\|U(\eta_0)p - q\|_2$. The main assumption in the analysis is that elements $\eta_0 \circ \gamma_i$ and $\eta_0^{-1} \circ \delta_i$ belong to $\mathcal{T}_d$.

We provide guarantees on the registration error when $d(p, q) < \epsilon\sqrt{\|c\|_2^2 + \|d\|_2^2}$, where ϵ accounts for the innovations between the patterns (other than a global transformation in $\mathcal{T}$). To this end, we first show that the notion of linear independence does not capture accurately the behaviour of our algorithm. Moreover, existing characterizations of the dictionary (e.g., RIP [3], coherence) are not well suited to the considered dictionary. Consequently, we introduce two novel properties on the dictionary.

Definition 1. *A dictionary $\mathcal{D}$ is (K, ϵ, α)-robustly linearly independent (RLI) if for any set $a \in \mathbb{R}^K$ and any subset $\{\phi_{\lambda_1}, \ldots, \phi_{\lambda_K}\}$ in $\mathcal{D}$, we have:*

$$\left\| \sum_{i=1}^{K} a_i \phi_{\lambda_i} \right\|_2 < \epsilon \|a\|_2$$
$$\implies \exists i, j \text{ with } a_i, a_j \neq 0, \left\| \frac{a_i \phi_{\lambda_i}}{\|a_i \phi_{\lambda_i}\|_2} + \frac{a_j \phi_{\lambda_j}}{\|a_j \phi_{\lambda_j}\|_2} \right\|_2 \leq \alpha. \quad (1)$$

In words, when ϵ and α are small, any linear combination of K atoms in $\mathcal{D}$ that nearly vanishes contains at least two atoms that approximately cancel each other. RLI can be seen as a weak form of RIP, where the norm of a linear combination of a set of K atoms is allowed to be close to zero provided that the condition in Eq. (1) is satisfied.

The second property compares the action of a transformation on two distinct atoms in the dictionary:

Definition 2. *The transformation inconsistency ρ of $\mathcal{D}$ is defined by:*

$$\rho = \sup_{\gamma, \gamma' \in \mathcal{T}_d} \sup_{\eta \in \mathcal{T} \setminus \{\mathbb{I}\}} \left\{ \frac{\|U(\eta)\phi_{\gamma'} - \phi_{\gamma'}\|_2}{\|U(\eta)\phi_{\gamma} - \phi_{\gamma}\|_2} \right\},$$

where $\mathbb{I}$ is the identity transformation. Note that $\rho \geq 1$ in general and that $\rho = 1$ when $\mathcal{T}$ is an abelian group. A large value of ρ (i.e., $\rho \gg 1$) means that there exist two atoms in the dictionary that behave very differently when they are subject to the same transformation.

Equipped with these novel dictionary properties, we propose a theorem that establishes an upper bound on the registration error:

Theorem 1. *If $d(p, q) < \epsilon\sqrt{\|c\|_2^2 + \|d\|_2^2}$ with $\epsilon > 0$, then:*

$$E(p, q) \leq \alpha\rho \min(\|c\|_1, \|d\|_1),$$

when $\mathcal{D}$ is $(2K, \epsilon, \alpha)$-RLI for some $\alpha \in [0, \sqrt{2})$, and ρ is the transformation inconsistency of $\mathcal{D}$.

Such a bound on the registration performance could not have been established using traditional characterizations of redundant dictionaries (e.g., coherence or RIP), since these quantities are very close to one when $\mathcal{T}_d$ is a dense discretization of $\mathcal{T}$. This performance bound has been shown to offer valuable insights in the analysis of our novel registration algorithm [2].

To the best of our knowledge, this paper constitutes the first theoretically motivated work for image registration through sparse representation in redundant dictionaries. It provides understanding of the dictionary properties that drive the registration performance.

References

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